

# PRIME AND PRIMITIVE KUMJIAN-PASK ALGEBRAS

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**ABSTRACT.** For a row-finite  $k$ -graph  $\Lambda$  with no sources and a unital commutative ring  $R$ , we give some conditions for  $\Lambda$  and  $R$  so that the Kumjian-Pask algebra  $KP_R(\Lambda)$  to be prime as well as primitive. Then the desourcifying method due to Farthing implies that our results may be generalized for every row-finite locally convex  $k$ -graphs. Moreover, by applying those results, we can determined prime graded basic ideals of Kumjian-Pask algebras as well as the primitive ones.

## 1. INTRODUCTION

For giving a generalization and graphical version of higher dimensional Cuntz-Krieger algebras [17], Kumjian and Pask in [9] introduced  $k$ -graphs and their relative  $C^*$ -algebras. The class of higher rank graph  $C^*$ -algebras naturally includes graph  $C^*$ -algebras and every (directed) graph may be considered as a 1-graph. Since the  $C^*$ -algebras of higher rank graphs have been significantly interested by  $C^*$ -algebraists, many structural results for graph  $C^*$ -algebras are extended in setting of higher rank graphs. (See [8, 9, 16] among others.)

A Leavitt path algebra  $L_R(E)$  is an algebraic analogue of graph  $C^*$ -algebras  $C^*(E)$ , that first introduced in [1] for a row-finite (directed) graph  $E$  over a field  $R$  and then it was extended in [2] and [18] for every graph  $E$  and unital commutative ring  $R$ . Leavitt path algebras are so called because they also generalize the algebras  $L(1, n)$  without invariant basis studied by Leavitt in [12]. Instead of similarities in definition, there are some differences between a graph  $C^*$ -algebra  $C^*(E)$  and the Leavitt path algebra  $L_R(E)$  that one should take notice:

- (1) The coefficients of  $C^*(E)$  belong to  $\mathbb{C}$ , but the coefficients of  $L_R(E)$  belong to a unital commutative ring  $R$ .
- (2) Many similar results for these classes of algebras have been obtained independently by different techniques so that each one may not be applied for the others.
- (3) Even in the case  $R = \mathbb{C}$ , the algebras  $C^*(E)$  and  $L_R(E)$  may have different structural properties. For example, if  $E$  is a graph with

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one vertex and one edge,  $C^*$ -algebra  $C^*(E)$  ( $= C(\mathbb{T})$ ) is not prime where as  $L_{\mathbb{C}}(E)$  ( $= \mathbb{C}[x, x^{-1}]$ ) is a prime ring. This causes that in many cases one could not simultaneously investigate both graph  $C^*$ -algebras and Leavitt path algebras.

Let  $\Lambda$  be a row-finite  $k$ -graph without sources and  $R$  be a unital commutative ring. Motivated in [4] the authors introduced a  $\mathbb{Z}^k$ -graded algebra  $KP_R(\Lambda)$  which is called the Kumjian-Pask algebra of  $\Lambda$  over  $R$  and proved two kinds of uniqueness theorems, called the graded uniqueness theorem and the Cuntz- Krieger uniqueness theorem. When  $R$  is a field, [4, Theorem 5.1] implies that every graded ideal of  $KP_R(\Lambda)$  is the form  $I_H$  which is generated by saturated and hereditary subset  $H$  of vertices. However, in general the structure of graded ideals are more complicated because it could depend not only on  $\Lambda$  but also on  $R$ . Since we usually prefer to focus only on  $\Lambda$  rather than  $\Lambda$  and  $R$ , we usually investigate basic graded ideals. Recall that an ideal  $I$  of  $KP_R(\Lambda)$  is called basic if  $rp_v \in I$  implies  $p_v \in I$  for every  $v \in \Lambda^0$  and  $r \in R \setminus \{0\}$ . Note that if  $R$  is a field, every ideal of  $KP_R(\Lambda)$  is naturally basic, so in this case the basic property is trivial.

More recently in [14] Kumjian-Pask algebras were defined for arbitrary row-finite  $k$ -graphs being “locally convex”. It is shown that the desourcifying method could extend the results of [4] for these algebras.

Recall that if  $E$  is a directed graph and  $R$  is a field, prime ideals as well as primitive ideals of  $L_R(E)$  are described in [11], via special subsets of the vertex set so-called maximal tails. First, the prime and primitive Leavitt-path algebras are determined by underlying graphs. So, prime graded ideals and primitive graded ideals of a Leavitt-path algebra are characterized by applying quotient graphs. Then it is shown that non graded ones are corresponding with prime ideals in  $K[x, x^{-1}]$  and special families of maximal tails. However, if  $R$  is not a field, the spaces of prime ideals and primitive ideals of  $L_R(E)$  are more complicated.

In this article, we study the structure of prime and primitive basic graded ideals of a Kumjian-Pask algebra  $KP_R(\Lambda)$ . We will allow the coefficient ring  $R$  to be an arbitrary unital commutative ring rather than a field and also  $\Lambda$  to be a locally convex row-finite  $k$ -graph. In Section 2, we review some basic definitions and results of higher rank graphs and Kumjian-Pask algebras of [4] and [14]. Then in Section 3, we provide some equivalent conditions for the primeness and the primitivity of Kumjian-Pask algebra  $KP_R(\Lambda)$  of a  $k$ -graph  $\Lambda$  with no sources. We extend these results for locally convex  $k$ -graphs in Section 4. For this, we first review some properties and results of desourcifying method of [16] in Section 4. Then by applying this method we characterize the prime and primitive Kumjian-Pask algebra of locally convex  $k$ -graphs. In particular, the prime and primitive basic graded ideals of a Kumjian-Pask algebra  $KP_R(\Lambda)$  are described. Also, when  $R$  is a field and  $\Lambda$  is strongly aperiodic, all prime and primitive ideals of  $KP_R(\Lambda)$  will be determined.

## 2. PRELIMINARIES

In this section, we recall some preliminaries which will be used in the next sections.

We denote the set of all natural numbers including zero by  $\mathbb{N}$ . Let  $k$  be a positive integer. For  $m, n \in \mathbb{N}^k$ , we write  $m \leq n$  if  $m_i \leq n_i$  for every  $1 \leq i \leq k$  and write  $m \vee n$  for the pointwise maximum of  $m$  and  $n$ . We usually denote the zero element  $(0, \dots, 0) \in \mathbb{N}^k$  by  $0$ .

*Definition 2.1.* For a positive integer  $k$ , a  $k$ -graph (or *higher rank graph*) is a countable category  $\Lambda = (\Lambda^0, \Lambda, r, s)$  equipped with a functor  $d : \Lambda \rightarrow \mathbb{N}^k$ , called the *degree map*, satisfying the following *factorization property*: for every  $\lambda \in \Lambda$  and  $m, n \in \mathbb{N}^k$  with  $d(\lambda) = m + n$ , there exist unique elements  $\mu, \nu \in \Lambda$  such that  $\lambda = \mu\nu$  and  $d(\mu) = m$ ,  $d(\nu) = n$ .

*Example 2.2.* For every ordinary directed graph  $E = (E^0, E^1, r, s)$ , the path category is naturally a 1-graph. In this 1-graph,  $\Lambda^0$  is the set of vertices  $E^0$ . The set of morphisms are the finite paths  $E^*$  of  $E$ . Also, the degree map  $d : E^* \rightarrow \mathbb{N}$  is defined by  $d(\mu) = |\mu|$  (the length of  $\mu$ ).

*Example 2.3.* Let  $\Omega_k^0 := \mathbb{N}^k$  and  $\Omega_k := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q\}$ . Also define  $r, s : \Omega_k \rightarrow \Omega_k^0$  and  $d : \Omega_k \rightarrow \mathbb{N}^k$  by  $s(p, q) := q$ ,  $r(p, q) := p$ ,  $d(p, q) := q - p$ . The paths  $(p, q)$  and  $(r, s)$  are composable if  $q = r$  and  $(p, q)(r, s) = (p, s)$ . By this definition  $(\Omega_k, r, s, d)$  is a  $k$ -graph. Also, for  $m \in \mathbb{N}^k$ , we can define  $\Omega_{k,m}^0 := \{p \in \mathbb{N}^k : p \leq m\}$  and  $\Omega_{k,m} := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\}$ . With the same definition for  $r, s$  and  $d$ ,  $(\Omega_{k,m}, r, s, d)$  is a  $k$ -graph.

For  $v, w \in \Lambda^0$  and  $n \in \mathbb{N}^k$ , we define  $v\Lambda^n = \{\mu \in \Lambda : s(\mu) = w, r(\mu) = v\}$  and  $\Lambda^n = \{\mu \in \Lambda : d(\mu) = n\}$ . Then by the factorization property we may identify  $\Lambda^0$  with the objects of  $\Lambda$  and so the elements of  $\Lambda^0$  are called *vertices*.

*Definition 2.4.* A  $k$ -graph  $\Lambda$  is *row-finite* if the set  $v\Lambda^n = \{\mu \in \Lambda^n : r(\mu) = v\}$  is finite for every  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ . Also  $\Lambda$  has *no sources* if  $v\Lambda^n \neq \emptyset$  for every  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ . A  $k$ -graph  $\Lambda$  is called *locally convex*, if for every  $v \in \Lambda^0$  and distinct  $i, j \in \{1, 2, \dots, k\}$ ,  $\lambda \in v\Lambda^{e_i}$  and  $\mu \in v\Lambda^{e_j}$ , the sets  $s(\lambda)\Lambda^{e_j}$  and  $s(\mu)\Lambda^{e_i}$  are nonempty. We see that every  $k$ -graph without any source is locally convex.

For  $n \in \mathbb{N}^k$ , we define  $\Lambda^{\leq n} = \{\lambda \in \Lambda : d(\lambda) \leq n, s(\lambda)\Lambda^{e_i} = \emptyset \text{ whenever } d(\lambda) + e_i \leq n\}$ . If  $\Lambda$  has no sources then  $\Lambda^{\leq n} = \Lambda^n$ .

*Definition 2.5.* Let  $\Lambda$  be a row-finite locally convex  $k$ -graph and  $m \in (\mathbb{N} \cup \{\infty\})^k$ . A degree preserving functor  $x : \Omega_{k,m} \rightarrow \Lambda$  is called a *boundary path* of degree  $m$  if for every  $p \in \mathbb{N}^k$  and  $i \in \{1, 2, \dots, k\}$ ,  $p \leq m$  and  $p_i = m_i$  imply that  $x(p)\Lambda^{e_i} = \emptyset$ . We denote the set of boundary paths by  $\Lambda^{\leq \infty}$ . If  $x \in \Lambda^{\leq \infty}$ , we usually call  $x(0) \in \Lambda^0$  the range of  $x$  and write it by  $r(x)$ . The degree of  $x$  is denoted by  $d(x)$ .

For every  $x \in \Lambda^{\leq \infty}$  and  $n \in \mathbb{N}^k$  with  $n \leq d(x)$ , there is a boundary path  $\sigma^n(x)$  of degree  $d(x) - n$  that defined by  $\sigma^n(x)(p, q) := x(p + n, q + n)$  for  $p \leq q \leq d(x) - n$ . Note that for every  $p \leq d(x)$ , we have  $x(0, p)\sigma^p(x) = x$ . Also, if  $\lambda \in \Lambda x(0)$ , there is a unique boundary path  $\lambda x : \Omega_{k, m+d(\lambda)} \rightarrow \Lambda$  such that  $\lambda x(0, d(\lambda)) = \lambda$ ,  $(\lambda x)(d(\lambda), d(\lambda) + p) = x(0, p)$ . If  $\Lambda$  has no sources, every boundary path is an infinite path; so, in this case, we denote  $\Lambda^\infty = \Lambda^{\leq \infty}$ .

*Definition 2.6.* Let  $\Lambda$  be a row-finite  $k$ -graph. we say that  $\Lambda$  is *aperiodic* if for every  $v \in \Lambda^0$  and each  $m \neq n \in \mathbb{N}^k$ , there is  $x \in \Lambda^{\leq \infty}$  such that either  $m - m \wedge d(x) \neq n - n \wedge d(x)$  or  $\sigma^{m \wedge d(x)}(x) \neq \sigma^{n \wedge d(x)}(x)$ . We say  $\Lambda$  is *periodic* if is not aperiodic.

For a row-finite  $k$ -graph  $\Lambda$  with no sources, every boundary path is an infinite path. So, for every  $m, n \in \mathbb{N}^k$  and  $x \in \Lambda^{\leq \infty}$ ,  $m \wedge d(x) = m$ ,  $n \wedge d(x) = n$ . Hence, when  $\Lambda$  has no sources, the definition of aperiodicity presented above is equivalent to the same definition given in [4].

Let  $\Lambda$  be a  $k$ -graph and  $\Lambda^{\neq 0} = \{\lambda \in \Lambda : d(\lambda) \neq 0\}$ . For every  $\lambda \in \Lambda$ , we define a *ghost path*  $\lambda^*$  and for  $v \in \Lambda^0$ , we define  $v^* := v$ . The set of ghost paths are denoted by  $G(\Lambda)$ . We extend  $d, r, s$  to  $G(\Lambda)$  by

$$d(\lambda^*) = -d(\lambda), \quad r(\lambda^*) = s(\lambda), \quad s(\lambda^*) = r(\lambda).$$

Also the composition on  $G(\Lambda)$  is defined by  $\lambda^* \mu^* = (\mu \lambda)^*$  for  $\mu, \lambda \in \Lambda^{\neq 0}$ .

*Definition 2.7.* Let  $\Lambda$  be a row-finite locally convex  $k$ -graph and  $R$  be a unital commutative ring. A *Kumjian-Pask  $\Lambda$ -family*  $(P, S)$  in an  $R$ -algebra  $A$  consists of two functions  $P : \Lambda^0 \rightarrow A$  and  $S : \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0}) \rightarrow A$  satisfying the following conditions.

- (KP1)  $\{P_v : v \in \Lambda^0\}$  is a family of mutually orthogonal idempotents.
- (KP2) for all  $\lambda, \mu \in \Lambda^{\neq 0}$  with  $r(\mu) = s(\lambda)$ , we have  $S_\lambda S_\mu = S_{\lambda\mu}$ ,  $S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}$  and

$$\begin{aligned} P_{r(\lambda)} S_\lambda &= S_\lambda P_{s(\lambda)} = S_\lambda, \\ P_{s(\lambda)} S_{\lambda^*} &= S_{\lambda^*} P_{r(\lambda)} = S_{\lambda^*}. \end{aligned}$$

- (KP3) for all  $\lambda, \mu \in \Lambda^{\neq 0}$  with  $d(\lambda) = d(\mu)$ , we have  $S_{\lambda^*} S_\mu = \delta_{\lambda, \mu} P_{s(\lambda)}$ .

- (KP4)  $P_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_{\lambda^*}$  whenever  $n \in \mathbb{N}^k \setminus \{0\}$  and  $v \in \Lambda^0$ .

For every locally convex  $k$ -graph  $\Lambda$ , [14, Theorem 3.7] implies that there is an  $R$ -algebra  $\text{KP}_R(\Lambda)$  generated by a Kumjian-Pask  $\Lambda$ -family  $(P, S)$  such that if  $(Q, T)$  is a Kumjian-Pask  $\Lambda$ -family in an  $R$ -algebra  $A$ , there exists an  $R$ -algebra homomorphism  $\pi_{Q, T} : \text{KP}_R(\Lambda) \rightarrow A$  such that  $\pi_{Q, T}(P_v) = Q_v$ ,  $\pi_{Q, T}(S_\lambda) = T_\lambda$ , and  $\pi_{Q, T}(S_{\lambda^*}) = T_{\lambda^*}$ . The  $R$ -algebra  $\text{KP}_R(\Lambda)$  is called the *Kumjian-Pask algebra of  $\Lambda$  with coefficients in  $R$* .

*Definition 2.8.* A ring  $R$  is called  $\mathbb{Z}^k$ -graded if there is a collection of additive subgroups  $\{R_n\}_{n \in \mathbb{Z}^k}$  of  $R$  such that  $R_{n_1} R_{n_2} \subseteq R_{n_1 + n_2}$  and every nonzero element  $a \in R$  can be written uniquely as finite sum  $\sum_{n \in G} a_n$  of nonzero

elements  $a_n \in R_n$ . The subgroup  $R_n$  is said the *homogeneous component of  $R$  of degree  $n$* . In this case, an ideal  $I$  of  $R$  is called *graded* if  $\{I \cap R_n : n \in \mathbb{Z}^k\}$  is a grading of  $I$ . Furthermore, if  $\phi : R \rightarrow S$  is a ring homomorphism between graded rings,  $\phi$  is called *graded* if  $\phi(R_n) \subseteq S_n$  for all  $n \in \mathbb{Z}^k$ . Note that the kernel of a graded homomorphism is always a graded ideal. If  $I$  is a graded ideal of ring  $R$ , then the quotient  $R/I$  is naturally graded with homogeneous components  $\{R_n + I\}_{n \in \mathbb{Z}^k}$  and the quotient map  $R \rightarrow R/I$  is a graded homomorphism.

By [14, Theorem 3.7], there is a  $\mathbb{Z}^k$ -grading on  $\text{KP}_R(\Lambda)$  satisfying

$$\text{KP}_R(\Lambda)_n = \text{span}_R\{s_\alpha s_{\beta^*} : \alpha, \beta \in \Lambda, d(\alpha) - d(\beta) = n\}.$$

### 3. PRIME AND PRIMITIVE KUMJIAN-PASK ALGEBRAS

Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and  $R$  be a unital commutative ring. In this section, we give some conditions for  $\Lambda$  and  $R$  such that  $\text{KP}_R(\Lambda)$  is a prime ring as well as primitive ring. We will generalize our results in Section 4 for row-finite locally convex  $k$ -graphs.

Recall that an ideal  $I$  of  $\text{KP}_R(\Lambda)$  is called *basic* if  $rp_v \in I$  with  $r \in R \setminus \{0\}$  implies  $p_v \in I$ . An ideal  $I$  of ring  $R$  is called *prime* if for each pair of ideals  $I_1, I_2$  of  $R$  with  $I_1 I_2 \subseteq I$ , at least one of them is contained in  $I$ . We say that a ring  $R$  is *prime* if the zero ideal of  $R$  is prime. Also, in a graded algebra, a graded ideal  $I$  is prime if and only if for every graded ideals  $I_1, I_2$  with  $I_1 I_2 \subseteq I$ , we have  $I_1 \subseteq I$  or  $I_2 \subseteq I$  [13, Proposition II.1.4]. We will use this fact in the proof of Theorem 3.3.

To give some necessary and sufficient conditions for the primeness of  $\text{KP}_R(\Lambda)$  in Theorem 3.3, we need the following lemma.

**Lemma 3.1.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and  $R$  be a unital commutative ring. If  $I$  is a nonzero graded ideal of  $\text{KP}_R(\Lambda)$ , then there exists  $rp_v \in I$  for some  $v \in \Lambda^0$  and  $r \in R \setminus \{0\}$ .*

*Proof.* Suppose that  $I$  is a graded ideal of  $\text{KP}_R(\Lambda)$ . Take an element  $x \in I_n = I \cap \text{KP}_R(\Lambda)_n$  for some  $n \in \mathbb{Z}^k$ . Since  $x \in \text{KP}_R(\Lambda)_n$ , we can write  $x = \sum_{i=1}^m r_i s_{\alpha_i} s_{\beta_i^*}$  such that  $d(\alpha_i) - d(\beta_i) = n$ , where  $\alpha_i, \beta_i \in \Lambda$  and  $r_i \in R \setminus \{0\}$ . Set  $t = \bigvee_{1 \leq i \leq m} d(\beta_i)$ . As  $\Lambda$  has no sources, for each  $1 \leq i \leq m$  apply (KP4) with  $t_i = t - d(\beta_i)$  to have

$$s_{\alpha_i} s_{\beta_i^*} = s_{\alpha_i} p_{s(\alpha_i)} s_{\beta_i^*} = \sum_{\lambda \in s(\alpha_i) \Lambda^{t_i}} s_{(\alpha_i \lambda)} s_{(\beta_i \lambda)^*}.$$

It is clear that  $d(\alpha_i \lambda) - d(\beta_i \lambda) = n$ . Hence, we can write

$$x = \sum_{i=1}^m r_i s_{\alpha_i} s_{\beta_i^*} = \sum_{i=1}^m \sum_{\lambda_i \in s(\alpha_i) \Lambda^{t_i}} r_i s_{(\alpha_i \lambda_i)} s_{(\beta_i \lambda_i)^*}$$

and all  $\beta_i \lambda_i$  have the same degree  $t$ . This yields that all  $\alpha_i \lambda_i$  also have the same degree  $n - t$  and so

$$s_{(\alpha_1 \lambda_1)^*} x s_{\beta_1 \lambda_1} = r_1 p_{s(\alpha_1 \lambda_1)} \in I$$

by (KP3).  $\square$

Let  $\Lambda$  be a row-finite locally convex  $k$ -graph. We define a relation on  $\Lambda^0$  by setting  $v \leq w$  if there is some  $\lambda \in v\Lambda w$ . A subset  $H$  of  $\Lambda^0$  is *hereditary* if  $v \in H$  and  $v\Lambda w \neq \emptyset$  imply  $w \in H$ . Also,  $H$  is *saturated* if  $\{s(\lambda) : \lambda \in v\Lambda^{\leq e_i}\} \subseteq H$  for some  $1 \leq i \leq k$  implies  $v \in H$ . So, for a row-finite  $k$ -graph  $\Lambda$  with no sources, the definition of saturation is equivalent to:  $s(v\Lambda^n) \subseteq H$  for some  $n \in \mathbb{N}^k$  implies  $v \in H$ . The saturation of  $H$ , denoted by  $\overline{H}$ , is the smallest saturated subset of  $\Lambda^0$  containing  $H$ . Recall that for a saturated and hereditary subset  $H$  of  $\Lambda^0$ ,

$$I_H = \text{span}_R\{s_\mu s_{\lambda^*} : s(\mu) = s(\lambda) \in H\}$$

is a basic and graded ideal of  $\text{KP}_R(\Lambda)$  [14, Theorem 9.4]. Also, for an ideal  $I$  of  $\text{KP}_R(\Lambda)$ , we define  $H_I := \{v \in \Lambda^0 : p_v \in I\}$ . From [4, Lemma 5.2],  $H_I$  is a saturated and hereditary subset of  $\Lambda^0$ .

**Definition 3.2.** Let  $\Lambda$  be a row-finite locally convex  $k$ -graph. A nonempty subset  $M$  of  $\Lambda^0$  is called *maximal tail* if  $M$  satisfies in the following conditions.

- (MT1) If  $w \in M$  and  $v \in \Lambda^0$  with  $v\Lambda w \neq \emptyset$ , then  $v \in M$ .
- (MT2) If  $v \in M$ , for every  $1 \leq i \leq k$  there exists  $\lambda \in v\Lambda^{\leq e_i}$  such that  $s(\lambda) \in M$ .
- (MT3) For  $v_1, v_2 \in M$ , there exists  $w \in M$  such that  $v_1\Lambda w \neq \emptyset$  and  $v_2\Lambda w \neq \emptyset$ .

Note that a subset  $H \subseteq \Lambda^0$  is hereditary and saturated if and only if  $\Lambda^0 \setminus H$  satisfies Conditions (MT1) and (MT2). In the following theorem, we characterize the prime Kumjian-Pask algebras when the underlying  $k$ -graphs have no sources. We will generalize it in Theorem 4.4 for locally convex  $k$ -graphs.

**Theorem 3.3.** Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and  $R$  be a unital commutative ring. Then the following are equivalent.

- (1)  $\text{KP}_R(\Lambda)$  is a prime ring.
- (2)  $R$  is an ID (Integral Domain) and  $\Lambda^0$  satisfies Condition (MT3).
- (3)  $R$  is an ID and  $H_I \cap H_J \neq \emptyset$  for every nonzero basic graded ideals  $I, J$  of  $\text{KP}_R(\Lambda)$ .

*Proof.* **1  $\Rightarrow$  2:** Suppose that  $\text{KP}_R(\Lambda)$  is a prime ring and  $v, w \in \Lambda^0$ . Then  $\text{KP}_R(\Lambda)p_v\text{KP}_R(\Lambda)$  and  $\text{KP}_R(\Lambda)p_w\text{KP}_R(\Lambda)$  are two nonzero ideals. By the primeness of  $\text{KP}_R(\Lambda)$ ,  $\text{KP}_R(\Lambda)p_w\text{KP}_R(\Lambda)p_v\text{KP}_R(\Lambda)$  is also nonzero and so is the corner  $p_w\text{KP}_R(\Lambda)p_v$ . Therefore, there are  $\alpha, \beta \in \Lambda$  such that  $p_ws_\alpha s_{\beta^*}p_v \neq 0$  and  $s(\alpha) = s(\beta) = z$ . Hence  $\alpha \in w\Lambda z$  and  $\beta \in v\Lambda z$  which means  $\Lambda^0$  satisfies Condition (MT3). Now we show that  $R$  is an ID. By contradiction, if there exist nonzero elements  $r_1, r_2 \in R$  such that  $r_1 r_2 = 0$ , then  $r_1\text{KP}_R(\Lambda)$  and  $r_2\text{KP}_R(\Lambda)$  are two nonzero ideals of  $\text{KP}_R(\Lambda)$ . But  $r_1\text{KP}_R(\Lambda)r_2\text{KP}_R(\Lambda) = r_1 r_2\text{KP}_R(\Lambda) = \{0\}$ , which contradicts the primeness of  $\text{KP}_R(\Lambda)$ .

**2  $\Rightarrow$  3:** Let  $I$  and  $J$  be two nonzero basic graded ideals of  $\text{KP}_R(\Lambda)$ . By Lemma 3.1, there are  $v, w \in \Lambda^0$  such that  $p_v \in I$  and  $p_w \in J$ . Condition (MT3) implies that there is  $z \in \Lambda^0$  such that  $v\Lambda z \neq \emptyset$  and  $w\Lambda z \neq \emptyset$  and so  $p_z \in I \cap J$  and  $H_I \cap H_J \neq \emptyset$ .

**3  $\Rightarrow$  1:** Let  $R$  be an ID and  $H_I \cap H_J \neq \emptyset$  for every nonzero basic graded ideals  $I, J$  of  $\text{KP}_R(\Lambda)$ . We show that the zero ideal of  $\text{KP}_R(\Lambda)$  is prime. Since the zero ideal is graded, it is sufficient to show that  $IJ \neq \{0\}$  for every nonzero graded ideals  $I, J$  of  $\text{KP}_R(\Lambda)$ . If  $I$  and  $J$  are such ideals, by Lemma 3.1 there are  $v_1, v_2 \in \Lambda^0$  and  $r_1, r_2 \in R \setminus \{0\}$  such that  $r_1 p_{v_1} \in I$  and  $r_2 p_{v_2} \in J$ . If we set  $H_i = \{z \in \Lambda^0 : v_i \Lambda z \neq \emptyset\}$  for  $i \in \{1, 2\}$ , it is clear that  $H_1$  and  $H_2$  are two hereditary subsets of  $\Lambda^0$ . Let  $I' = I_{\overline{H_1}}$  and  $J' = I_{\overline{H_2}}$ . Then  $p_{v_1} \in I'$ ,  $p_{v_2} \in J'$  and we have  $r_1 I' \subseteq I$  and  $r_2 J' \subseteq J$ . Since  $I', J'$  are basic graded ideals of  $\text{KP}_R(\Lambda)$ , from the statement (3) we get  $\overline{H_1} \cap \overline{H_2} \neq \emptyset$ . Thus, there is  $z_0 \in \overline{H_1} \cap \overline{H_2}$  and since  $R$  is an ID, we have  $r_1 r_2 \neq 0$ ,  $r_1 r_2 p_{z_0} \in IJ$ , and so  $IJ \neq \{0\}$ .  $\square$

Now we consider primitive Kumjian-Pask algebras. Recall that a ring  $R$  is called *left primitive* (*right primitive*) if it has a faithful simple left (right)  $R$ -module. Since a Kumjian-Pask algebra is left primitive if and only if it is right primitive, in this case, we simply say it is *primitive*. Note that every primitive ring is prime. Also, every commutative primitive ring is a field.

**Lemma 3.4** ([10, Lemmas 2.1 and 2.2]). *Let  $R$  be a field and  $R_1$  be a prime  $R$ -algebra. Then there exists a prime unital  $R$ -algebra  $R_2$  which  $R_1$  embeds in  $R_2$  as an ideal. Furthermore,  $R_2$  is primitive if and only if  $R_1$  is primitive.*

**Lemma 3.5** ([7, Theorem 1]). *A unital ring  $R$  is left primitive if and only if there is a left ideal  $M \neq R$  of  $R$  such that for every nonzero two sided ideal  $I$  of  $R$ , we have  $M + I = R$ .*

We use the infinite-path representation of a  $k$ -graph defined in [4, Section 3] to prove Theorem 3.7 below.

**Definition 3.6** ([4]). Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and  $R$  be a unital commutative ring. For  $v \in \Lambda^0$  and  $\mu, \nu \in \Lambda^{\neq 0}$ , we define the maps  $f_v, f_\lambda, f_\mu^* : \Lambda^\infty \rightarrow \mathbb{F}_R(\Lambda^\infty)$  by

$$\begin{aligned} f_v(x) &= \begin{cases} x & \text{if } x(0) = v \\ 0 & \text{otherwise,} \end{cases} \\ f_\lambda(x) &= \begin{cases} \lambda x & \text{if } x(0) = s(\lambda) \\ 0 & \text{otherwise, and} \end{cases} \\ f_\mu^*(x) &= \begin{cases} x(d(\mu), \infty) & \text{if } x(0, d(\mu)) = \mu \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where  $\mathbb{F}_R(\Lambda^\infty)$  is the free module with basis the infinite path space. By the universal property of free modules, there are nonzero endomorphisms  $Q_v, T_\lambda, T_\mu^* : \mathbb{F}_R(\Lambda^\infty) \rightarrow \mathbb{F}_R(\Lambda^\infty)$  extending  $f_v, f_\lambda, f_\mu^*$ . In [4], it is shown



that  $(Q, T)$  is a Kumjian-Pask  $\Lambda$ -family in  $\text{End}(\mathbb{F}_R(\Lambda^\infty))$ . So there is an  $R$ -algebra homomorphism  $\pi_{Q,T} : \text{KP}_R(\Lambda) \rightarrow \text{End}(\mathbb{F}_R(\Lambda^\infty))$  such that  $\pi_{Q,T}(p_v) = Q_v$ ,  $\pi_{Q,T}(s_\lambda) = T_\lambda$ , and  $\pi_{Q,T}(s_{\mu^*}) = T_{\mu^*}$ . The homomorphism  $\pi_{Q,T}$  is called the *infinite-path representation* of  $\text{KP}_R(\Lambda)$ .

Recall from Corollary 4.10 and Lemma 5.9 of [4] that the infinite-path representation  $\pi_{Q,T}$  of  $\text{KP}_R(\Lambda)$  is faithful if and only if  $\Lambda$  is aperiodic.

**Theorem 3.7.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and  $R$  be a unital commutative ring. Then  $\text{KP}_R(\Lambda)$  is primitive if and only if*

- (1)  $\Lambda^0$  satisfies Condition (MT3),
- (2)  $\Lambda$  is aperiodic, and
- (3)  $R$  is a field.

*Proof.* First assume that the above three conditions hold. By Theorem 3.3,  $\text{KP}_R(\Lambda)$  is a prime ring. Lemma 3.4 yields that there exists a prime unital ring  $R_2$  such that  $\text{KP}_R(\Lambda)$  embeds in  $R_2$  as an ideal and primitivity of them are equivalent. So, it suffices to prove  $R_2$  is a primitive ring. Taking an arbitrary vertex  $v \in \Lambda^0$ , suppose that  $H = \{w \in \Lambda^0 : v \leq w\}$  and write  $H = \{v_1, v_2, \dots\}$ . We claim that there exists a sequence  $\{\lambda_i\}_{i=1}^\infty$  of paths in  $\Lambda$  such that for every  $i \in \mathbb{N}$ ,  $\lambda_{i+1} = \lambda_i \mu_i$  for some path  $\mu_i \in \Lambda$  and also  $v_i \leq s(\lambda_i)$ . For this, set  $\lambda_1 = v_1$ . Clearly,  $\lambda_1$  satisfies the properties. Assume that there are  $\lambda_1, \dots, \lambda_n$  with the indicated properties. By Condition (MT3), there is  $u \in \Lambda^0$  such that  $v_{n+1} \leq u$  and  $s(\lambda_n) \leq u$  and so there is  $\mu \in \Lambda$  with  $s(\mu) = u$  and  $r(\mu) = s(\lambda_n)$ . If we set  $\lambda_{n+1} := \lambda_n \mu$ , then  $\lambda_{n+1}$  has the desired properties. Hence, the claim holds. Note that since each  $\lambda_i$  is a subpath of  $\lambda_{i+1}$ , for every  $n > i$ , we have

$$s_{\lambda_i} s_{\lambda_i}^* s_{\lambda_n} s_{\lambda_n}^* = s_{\lambda_n} s_{\lambda_n}^*.$$

Define  $M = \sum_{i=1}^\infty R_2(1 - s_{\lambda_i} s_{\lambda_i}^*)$  that is a left ideal of  $R_2$ . We see that  $M \neq R_2$ . Indeed, if  $M = R_2$ , we have  $1 \in M$ . So, there are  $r_1, \dots, r_m \in R_2$  such that  $1 = \sum_{i=1}^m r_i(1 - s_{\lambda_i} s_{\lambda_i}^*)$  which follows

$$s_{\lambda_m} s_{\lambda_m}^* = \sum_{i=1}^m r_i(1 - s_{\lambda_i} s_{\lambda_i}^*) s_{\lambda_m} s_{\lambda_m}^* = 0,$$

a contradiction. Therefore,  $1 \notin M$  and  $M \neq R_2$ .

Now suppose that  $I$  is an arbitrary two sided ideal of  $R_2$ . We show that  $M + I = R_2$ . Since  $R_2$  is prime and  $\text{KP}_R(\Lambda)$  is a two sided ideal of  $R_2$ , we have  $I_1 = I \cap \text{KP}_R(\Lambda)$  is a nonzero two sided ideal of  $\text{KP}_R(\Lambda)$ . Since  $\Lambda$  is aperiodic, an application of the Cuntz-Krieger uniqueness theorem implies that  $I_1$  contains a vertex idempotent  $p_w$  (see [4, Proposition 5.11]). By Condition (MT3), there exists  $z \in \Lambda^0$  such that  $w \leq z$  and  $v \leq z$ . So,  $z = v_n$  for some  $n \geq 1$ . This yields that  $p_{s(\lambda_n)} \in I$  and  $s_{\lambda_n} s_{\lambda_n}^* \in I$ . As  $1 = (1 - s_{\lambda_n} s_{\lambda_n}^*) + s_{\lambda_n} s_{\lambda_n}^*$ , we get  $1 \in M + I$  and  $M + I = R_2$ . Therefore, the left ideal  $M$  satisfies the conditions of Lemma 3.5 and so  $R_2$  is primitive. By Lemma 3.4, we conclude that  $\text{KP}_R(\Lambda)$  is primitive.



Conversely, suppose that  $\text{KP}_R(\Lambda)$  is primitive. Therefore,  $\text{KP}_R(\Lambda)$  is a prime ring and  $\Lambda^0$  satisfies Condition (MT3) by Theorem 3.3. We show that  $\Lambda$  is aperiodic. Since  $\text{KP}_R(\Lambda)$  is a primitive ring, it has a faithful simple left  $\text{KP}_R(\Lambda)$ -module  $M$ . Since  $M$  is simple, there is a maximal left ideal  $J$  of  $\text{KP}_R(\Lambda)$  such that  $M \cong \text{KP}_R(\Lambda)/J$  as modules. This implies that for every  $a \in J$  we have  $aM = 0$ . Since  $M$  is faithful, we get  $a = 0$  for every  $a \in J$ . So  $J = 0$ ,  $M \cong \text{KP}_R(\Lambda)$ , and  $\text{KP}_R(\Lambda)$  is a simple  $\text{KP}_R(\Lambda)$ -module. If  $\Lambda$  is periodic, [4, Lemma 5.9] implies that the kernel of infinite-path representation  $\pi_{Q,T}$  is a nonzero ideal of  $\text{KP}_R(\Lambda)$ . Also,  $\pi_{Q,T}$  contains no vertex idempotents [4, Proposition 5.11] and so,  $\ker \pi_{Q,T}$  is a proper nonzero (two sided) ideal of  $\text{KP}_R(\Lambda)$ . This contradicts the simplicity of  $\text{KP}_R(\Lambda)$ , and hence,  $\Lambda$  is aperiodic.

It remains to show  $R$  is a field. If  $M$  is a simple and faithful left  $\text{KP}_R(\Lambda)$ -module, then there is a maximal left ideal  $J$  of  $\text{KP}_R(\Lambda)$  such that  $M \cong \text{KP}_R(\Lambda)/J$ . It follows that for every  $a \in J$ , we have  $aM = 0$  and so  $a = 0$  by the faithfulness of  $M$ . Thus  $J = 0$  and  $M \cong \text{KP}_R(\Lambda)$ . If  $R$  is not a field, there is a nonzero proper ideal  $I$  of  $R$ . Then  $IM \cong I\text{KP}_R(\Lambda)$  is a nonzero proper left  $\text{KP}_R(\Lambda)$ -submodule of  $M$ . This contradicts the simplicity of  $M$ , and therefore,  $R$  must be a field.  $\square$

#### 4. LOCALLY CONVEX $k$ -GRAPHS

Let  $\Lambda$  be a row-finite locally convex  $k$ -graph. In [14, Theorem 7.4], it is shown that there is a row-finite  $k$ -graph  $\tilde{\Lambda}$  with no sources such that  $\text{KP}_R(\Lambda)$  and  $\text{KP}_R(\tilde{\Lambda})$  are Morita equivalent. In this section, we first review the construction of  $\tilde{\Lambda}$  due to Farthing [6] and then generalize Theorems 3.3 and 3.7 for row-finite locally convex  $k$ -graphs.

For a row-finite locally convex  $k$ -graph  $\Lambda$ , define the sets  $V_\Lambda$  and  $P_\Lambda$  as

$$V_\Lambda := \{(x; m) : x \in \Lambda^{\leq \infty}, m \in \mathbb{N}^k\},$$

$$P_\Lambda := \{(x; (m, n)) : x \in \Lambda^{\leq \infty}, m \leq n \in \mathbb{N}^k\}.$$

If we define  $(x; m) \approx (y; n)$  if and only if

- V1)  $x(m \wedge d(x)) = y(n \wedge d(y))$  and
- V2)  $m - m \wedge d(x) = n - n \wedge d(y)$ ,

then  $\approx$  is an equivalence relation on  $V_\Lambda$  and we denote the class of  $(x; m)$  by  $[x; m]$ . Also, the relation  $(x; (m, n)) \sim (y; (p, q))$  if and only if

- P1)  $x(m \wedge d(x), n \wedge d(x)) = y(p \wedge d(y), q \wedge d(y))$ ,
- P2)  $m - m \wedge d(x) = p - p \wedge d(y)$ , and
- P3)  $n - m = q - p$

is an equivalence relation on  $P_\Lambda$  and we denote the equivalence class of  $(x; (m, n))$  by  $[x; (m, n)]$ .

Following [6], there is a  $k$ -graph  $\tilde{\Lambda}$  with  $\tilde{\Lambda}^0 := V_\Lambda / \approx$  and  $\tilde{\Lambda} := P_\Lambda / \sim$  such that

$$r([x; (m, n)]) = [x; m] \text{ and } s([x; (m, n)]) = [x; n],$$

$$\begin{aligned} [x; (m, n)] \circ [y; (p, q)] &= [x(0, n \wedge d(x))\sigma^{p \wedge d(y)}; (m, n + q - p)], \\ d([x; m]) &= 0 \text{ and } d([x; (m, n)]) = n - m. \end{aligned}$$

The  $k$ -graph  $\tilde{\Lambda} = (\tilde{\Lambda}^0, \tilde{\Lambda}, r, s, d)$  contains no sources which is called the *desourcification* of  $\Lambda$ . Recall from [14, Corollary 7.5] that there is a surjective Morita context between  $\text{KP}_R(\tilde{\Lambda})$  and  $\text{KP}_R(\Lambda)$ . Note that, by [4, Theorem 7.4], the primeness and primitivity are invariant under Morita contexts.

*Remark 4.1.* We may simply check that the paths  $(\lambda x, (0, d(\lambda)))$  and  $(\lambda y, (0, d(\lambda)))$  are equivalent under  $\sim$  for any  $\lambda \in \Lambda$  and  $x, y \in s(\lambda)\Lambda^{\leq \infty}$ . So, the map  $\iota : \Lambda \rightarrow \tilde{\Lambda}$  satisfying  $\iota(\lambda) = [\lambda x; (0, d(\lambda))]$  for any  $x \in s(\lambda)\Lambda^{\leq \infty}$  is a well-defined map. Indeed,  $\iota$  is an injective  $k$ -graph morphism and  $\Lambda$  and  $\iota(\Lambda)$  are isomorphic. Moreover, the map  $\pi : \tilde{\Lambda} \rightarrow \iota(\Lambda)$  defined by  $\pi[x; (m, n)] = [x; (m \wedge d(x), n \wedge d(x))]$  for  $x \in \Lambda^{\leq \infty}$  and  $m \leq n \in \mathbb{N}^k$  is a well-defined surjective  $k$ -graph morphism such that  $\pi \circ \pi = \pi$  and  $\pi \circ \iota = \text{id}$ . See [14, Section 6] for more details.

**Lemma 4.2** ([16, Lemma 2.2]). *Let  $\Lambda$  be a row-finite locally convex  $k$ -graph and  $v \in \Lambda^0$ . Suppose  $x \in v\Lambda^{\leq \infty}$  and  $p \in \mathbb{N}^k$  satisfying  $p \wedge d(x) = 0$ . Then for any other  $z \in v\Lambda^{\leq \infty}$ , we have  $p \wedge d(z) = 0$  and  $[x; (0, p)] = [z; (0, p)]$ .*

**Lemma 4.3.** *Let  $\Lambda$  be a row-finite locally convex  $k$ -graph and  $\tilde{\Lambda}$  be its desourcification. Then*

- (1)  $\tilde{\Lambda}$  is aperiodic if and only if  $\Lambda$  is.
- (2)  $\Lambda^0$  satisfies Condition (MT3) if and only if  $\tilde{\Lambda}^0$  so does.

*Proof.* The statement (1) is [16, Proposition 3.6]. For (2), suppose that  $\Lambda^0$  satisfies Condition (MT3) and take arbitrary vertices  $[x; m], [y; n] \in \tilde{\Lambda}^0$ . Without loss of generality, we may assume that  $m \wedge d(x) = n \wedge d(y) = 0$ . Since  $x(0), y(0)$  belong to  $\Lambda^0$ , there are  $\lambda \in x(0)\Lambda$  and  $\mu \in y(0)\Lambda$  such that  $s(\lambda) = s(\mu)$ . Take some  $z \in s(\lambda)\Lambda^{\leq \infty}$ . Then  $\lambda z \in x(0)\Lambda^{\leq \infty}$ ,  $\mu z \in y(0)\Lambda^{\leq \infty}$ , and by Lemma 4.2 we have  $[x; (0, m)] = [\lambda z; (0, m)]$  and  $[y; (0, n)] = [\mu z; (0, n)]$ . If  $t := (m - d(\lambda)) \vee (n - d(\mu)) \vee 0$ , then we have

$$\begin{aligned} r([\lambda z; (m, t + d(\lambda))]) &= [\lambda z; m] = [x; m], \\ r([\mu z; (n, t + d(\mu))]) &= [\mu z; n] = [y; n], \\ s([\lambda z; (m, t + d(\lambda))]) &= [\lambda z; t + d(\lambda)] = [z; t], \text{ and} \\ s([\mu z; (n, t + d(\mu))]) &= [\mu z; t + d(\mu)] = [z; t]. \end{aligned}$$

Therefore,  $[\lambda z; (m, t + d(\lambda))] \in [x; m]\tilde{\Lambda}$  and  $[\mu z; (n, t + d(\mu))] \in [y; n]\tilde{\Lambda}$  with the same sources. This says that  $\tilde{\Lambda}^0$  satisfies Condition (MT3), as desired.

Conversely, assume that  $\tilde{\Lambda}^0$  satisfies Condition (MT3). If  $v, w \in \Lambda^0$ , there are  $\lambda \in v\tilde{\Lambda}$  and  $\mu \in w\tilde{\Lambda}$  such that  $s(\lambda) = s(\mu)$ . So,  $\pi(\lambda), \pi(\mu) \in \Lambda$  and we have

$$r(\pi(\lambda)) = \pi(r(\lambda)) = \pi(v) = v,$$

$$\begin{aligned} r(\pi(\mu)) &= \pi(r(\mu)) = \pi(w) = w, \text{ and} \\ s(\pi(\lambda)) &= \pi(s(\lambda)) = \pi(s(\mu)) = s(\pi(\mu)). \end{aligned}$$

Thus  $\pi(\lambda) \in v\Lambda$ ,  $\pi(\mu) \in w\Lambda$  and  $s(\pi(\lambda)) = s(\pi(\mu))$ . This implies that  $\Lambda^0$  satisfies Condition (MT3).  $\square$

**Theorem 4.4.** *Let  $\Lambda$  be a row-finite locally convex  $k$ -graph and  $R$  be a unital commutative ring. Then  $\text{KP}_R(\Lambda)$  is prime if and only if  $R$  is an ID and  $\Lambda^0$  satisfies Condition (MT3).*

*Proof.* Let  $\text{KP}_R(\Lambda)$  be a prime ring. [14, Corollary 7.5] yields that there is a surjective Morita context between  $\text{KP}_R(\Lambda)$  and  $\text{KP}_R(\tilde{\Lambda})$ . As the primeness is preserved under surjective Morita contexts,  $\text{KP}_R(\tilde{\Lambda})$  is also a prime ring. Since  $\tilde{\Lambda}$  is a row-finite  $k$ -graph with no sources, Theorem 3.3 implies that  $R$  is an ID and  $\tilde{\Lambda}^0$  satisfies Condition (MT3). Hence,  $\Lambda^0$  satisfies Condition (MT3) either by Lemma 4.3.

For the converse, let  $\Lambda^0$  satisfy Condition (MT3) and  $R$  be an ID. As  $\tilde{\Lambda}^0$  satisfies Condition (MT3) by Lemma 4.3, we see that  $\text{KP}_R(\tilde{\Lambda})$  is a prime ring by applying Theorem 3.3. Now the Morita equivalence between  $\text{KP}_R(\tilde{\Lambda})$  and  $\text{KP}_R(\Lambda)$  gives the result.  $\square$

Now, we may characterize prime basic graded ideals in Kumjian-Pask algebra  $\text{KP}_R(\Lambda)$ , when  $\Lambda$  is a row-finite locally convex  $k$ -graph. Recall from [14, Theorem 9.4] that every basic graded ideal of  $\text{KP}_R(\Lambda)$  is of the form  $I_H$  for a saturated hereditary subset  $H$  of  $\Lambda^0$ . Note that when  $R$  is a field, every ideal of  $\text{KP}_R(\Lambda)$  is basic. We first state the following lemma.

**Lemma 4.5.** *Let  $\Lambda$  be a row-finite locally convex  $k$ -graph and  $R$  be a unital commutative ring. For a saturated and hereditary subset  $H$  of  $\Lambda^0$ ,  $\Lambda \setminus H = (\Lambda^0 \setminus H, s^{-1}(\Lambda^0 \setminus H), r, s)$  is a row-finite locally convex  $k$ -graph and  $\text{KP}_R(\Lambda \setminus H)$  is canonically isomorphic to  $\text{KP}_R(\Lambda)/I_H$ .*

*Proof.* First by [15, Theorem 5.2],  $\Lambda \setminus H$  is a row-finite locally convex  $k$ -graph. Then, similar to the proof of [4, Theorem 5.5], we may define an isomorphism between  $\text{KP}_R(\Lambda \setminus H)$  and  $\text{KP}_R(\Lambda)/I_H$ .  $\square$

We usually say a locally convex  $k$ -graph  $\Lambda$  to be *strongly aperiodic* if  $\Lambda \setminus H$  is aperiodic for every hereditary saturated subset  $H$  of  $\Lambda^0$ . When  $\Lambda$  is strongly aperiodic with no sources, [4, Corollary 5.7] follows that every ideal of  $\text{KP}_R(\Lambda)$  is of the form  $I_H$  for some saturated hereditary set  $H$ . Similarly, we may use [14, Theorem 9.4] to have the same result for  $\text{KP}_R(\Lambda)$  when  $\Lambda$  is a locally convex  $k$ -graph.

**Proposition 4.6.** *Let  $\Lambda$  be a row-finite locally convex  $k$ -graph. A basic graded ideal  $I_H$  of  $\text{KP}_R(\Lambda)$  is prime if and only if  $\Lambda^0 \setminus H$  is a maximal tail and  $R$  is an ID. In particular,*

- (1) *if  $R$  is a field, every prime graded ideal of  $\text{KP}_R(\Lambda)$  is of the form  $I_H$ , where  $\Lambda^0 \setminus H$  is a maximal tail;*

- (2) if  $R$  is a field and  $\Lambda$  is strongly aperiodic, every prime ideal of  $\text{KP}_R(\Lambda)$  is of the form  $I_H$ , where  $\Lambda^0 \setminus H$  is a maximal tail.

*Proof.* Let  $I_H$  be a prime ideal of  $\text{KP}_R(\Lambda)$ . Since  $\text{KP}_R(\Lambda)/I_H \cong \text{KP}_R(\Lambda \setminus H)$  by Lemma 4.5,  $\text{KP}_R(\Lambda \setminus H)$  is a prime ring. Therefore, Theorem 4.4 implies that  $R$  is an ID and  $\Lambda^0 \setminus H$  satisfies Condition (MT3). Clearly,  $\Lambda^0 \setminus H$  also satisfies Conditions (MT1) and (MT2) because  $H$  is hereditary and saturated. Hence,  $\Lambda^0 \setminus H$  is a maximal tail.

Conversely, suppose that  $\Lambda^0 \setminus H$  is a maximal tail and  $R$  be an ID. By Theorem 4.4,  $\text{KP}_R(\Lambda \setminus H)$  is a prime ring. Since  $\text{KP}_R(\Lambda)/I_H \cong \text{KP}_R(\Lambda \setminus H)$ , we conclude that  $I_H$  is a prime ideal of  $\text{KP}_R(\Lambda)$ .  $\square$

In Theorem 3.7, we characterize the primitive Kumjian-Pask algebras when the underlying  $k$ -graphs have no sources. We may apply desourcifying method to obtain a same result for row-finite locally convex  $k$ -graphs.

**Theorem 4.7.** *Let  $\Lambda$  be a row-finite locally convex  $k$ -graph and  $R$  be a unital commutative ring. Then  $\text{KP}_R(\Lambda)$  is primitive if and only if*

- 1)  $\Lambda^0$  satisfies Condition (MT3),
- 2)  $\Lambda$  is aperiodic, and
- 3)  $R$  is a field.

*Proof.* Let  $\text{KP}_R(\Lambda)$  be a primitive ring. By [14, Corollary 7.5], there is surjective Morita context between  $\text{KP}_R(\Lambda)$  and  $\text{KP}_R(\tilde{\Lambda})$ . As the primitivity is preserved under surjective Morita contexts,  $\text{KP}_R(\tilde{\Lambda})$  is also a primitive ring. Theorem 3.7 implies that  $\tilde{\Lambda}^0$  satisfies Condition (MT3),  $\tilde{\Lambda}$  is aperiodic and  $R$  is a field. Therefore,  $\Lambda^0$  satisfies Condition (MT3) and  $\Lambda$  is aperiodic by Lemma 4.3.

Conversely, let the above three conditions hold. By Lemma 4.3,  $\tilde{\Lambda}^0$  satisfies Condition (MT3) and  $\tilde{\Lambda}$  is aperiodic. So,  $\text{KP}_R(\tilde{\Lambda})$  is a primitive ring. By Theorem 3.3, the Morita equivalence between  $\text{KP}_R(\tilde{\Lambda})$  and  $\text{KP}_R(\Lambda)$  implies that  $\text{KP}_R(\Lambda)$  is also a primitive ring.  $\square$

**Proposition 4.8.** *Let  $\Lambda$  be a row-finite locally convex  $k$ -graph and  $R$  be a field. A graded ideal  $I_H$  of  $\text{KP}_R(\Lambda)$  is primitive if and only if the  $k$ -graph  $\Lambda \setminus H$  is aperiodic and  $\Lambda^0 \setminus H$  is a maximal tail. In particular, if  $\Lambda$  is strongly aperiodic, then every primitive ideal of  $\text{KP}_R(\Lambda)$  is of the form  $I_H$ , where  $\Lambda^0 \setminus H$  is a maximal tail.*

*Proof.* Let  $I_H$  be a primitive ideal of  $\text{KP}_R(\Lambda)$ . Since Lemma 4.5 yields that  $\text{KP}_R(\Lambda)/I_H \cong \text{KP}_R(\Lambda \setminus H)$  as rings,  $\text{KP}_R(\Lambda \setminus H)$  is a primitive ring. Therefore, Theorem 4.7 implies that  $\Lambda^0 \setminus H$  satisfies Condition (MT3) and  $\Lambda \setminus H$  is aperiodic.

Conversely, let  $\Lambda^0 \setminus H$  satisfy Condition (MT3) and  $\Lambda \setminus H$  be aperiodic. By Theorem 4.7, the Kumjian-Pask algebra  $\text{KP}_R(\Lambda \setminus H)$  is a primitive ring. Since  $\text{KP}_R(\Lambda)/I_H \cong \text{KP}_R(\Lambda \setminus H)$ , we conclude that  $I_H$  is a primitive ideal of  $\text{KP}_R(\Lambda)$ .  $\square$

Considering Propositions 4.6 and 4.8, we have the following result.

**Corollary 4.9.** *Let  $R$  be a field and  $\Lambda$  be a row-finite locally convex  $k$ -graph that is strongly aperiodic. Then every prime ideal of  $KP_R(\Lambda)$  is primitive and vice versa.*

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